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# More D-branes in the Nappi-Witten background

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Abstract: We re-examine the problem of determining the possible $D$-branes in the Nappi-Witten background. In addition to the known branes, we find that there are also $D$-instantons, flat euclidean $D$-strings and curved $D$-membranes admitting parallel spinors, all of which can be interpreted as (twisted) conjugacy classes in the Nappi-Witten group.

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## Contents





# 1. Introduction and motivation

*D*-branes have played a central role in many of the fascinating developments in string theory in the last few years. They have proved particularly versatile in bridging the gap between gauge theory and gravity. This is due to the fact that they admit two very different descriptions: as stable solutions of type-II supergravity on the one hand, and as boundary conditions for open strings on the other. Despite the fact that both descriptions play equally important roles in the gravity/gauge theory correspondence, the present state of our knowledge displays a conspicuous lack of symmetry. Whereas there has been much progress in constructing *D*-brane-type solutions to type II supergravity, it is only in very few cases that we can actually ascertain that these solutions describe possible boundary conditions for open strings. In particular, very little is known about *D*-branes (in the sense of open string boundary conditions) in nontrivial backgrounds, e.g., in curved space. This mirrors the fact that whereas one can write down many supergravity vacua, it is not clear how to describe string propagation on many of them. It therefore might seem unreasonable to expect a stringy description of a $D$-brane solution on a background for which string propagation (without branes) cannot be adequately described. By the same token, it is not unreasonable to expect that we should be able to understand $D$-branes in those backgrounds for which a conformal field theory can be written down. Such backgrounds include flat spaces, orbifolds, toroidal and Calabi-Yau compactifications, and WZW and coset models. In many of these cases, $D$-branes are fairly well understood from both geometric and conformal field theoretic points of view, but in some cases (e.g., the WZW model) a lot less is known than one might expect.

The purpose of the present paper is to re-examine the problem of determining the possible conformal invariant Dirichlet boundary conditions in a WZW model from a geometric perspective, and to illustrate the results in the case of the WZW model associated to the Nappi-Witten group [4]. This string background is privileged in that it can be treated exactly as a conformal field theory, and the geometry is also simple enough to be studied classically. It also displays, as we shall see, many of the features of $D$-branes present in general WZW models. The analysis complements the work in [7] in that we study a different type of gluing conditions for the chiral currents.

There is growing body of literature on the subject of boundary conditions in WZW models, and we will not attempt to list all relevant papers here, except for those which have a direct relation to the present one. A fuller comparative discussion of the literature can be found in [5], which explains the geometric interpretation of the gluing conditions on which this paper is based.

This note is organised as follows. In section 2 we discuss boundary conditions for WZW models associated to conjugacy classes and twisted generalisations thereof. In section 3 we illustrate these results to the Nappi-Witten group and we determine its (twisted) conjugacy classes, identifying those which can be interpreted as $D$-branes. We will see that among these classes one finds curved membranes, flat euclidean strings and instantons. In section 4 we summarise the results of the paper.

## 2. $D$-branes in WZW models

The WZW model is the theory of harmonic maps $g : \Sigma \to G$ from a two-dimensional worldsheet $\Sigma$ to a Lie group $G$ possessing a bi-invariant metric. This induces on the Lie algebra $\mathfrak{g}$ of $G$ an invariant scalar product. Let the worldsheet have boundary $\partial\Sigma$. In the simplest case, $D$-branes will correspond to certain submanifolds $B \subset G$ which can be used as boundary conditions; that is, such that the map $g$ takes the boundary of the worldsheet to $B$. In other words, in the presence of a $D$-brane, admissible field configurations are those maps $g : \Sigma \to G$ such that $g : \partial\Sigma \to B$. Not all submanifolds $B \subset G$ are allowed: a consistent boundary condition must



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preserve conformal invariance. Of course, there are more consistency conditions, namely the ones coming from demanding the compatibility between the open and closed string pictures for processes involving these $D$-branes. In order to impose these extra conditions, however, it is necessary to have the quantum field theory associated to the WZW under control, and in particular to know the spectrum. In this paper we will only focus on the consistency conditions coming from conformal invariance, or less generally from invariance under the current algebra.

The conformal symmetry of the WZW model is a consequence of the affine symmetry

$$g(z,\bar{z}) \mapsto \Omega(z) g(z,\bar{z}) \bar{\Omega}(\bar{z})^{-1}\,,$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are independent maps from $\Sigma$ to $G$ depending holomorphically and anti-holomorphically, respectively, on the complex coordinate. One way to guarantee conformal invariance of a boundary condition is therefore to preserve a large enough subgroup of the affine symmetry.

Let us fix one boundary component and choose a local parametrisation of the worldsheet for which the boundary component in question coincides with the real axis $z = \bar{z}$. Then at the boundary the affine symmetry becomes

$$g\big|_{\partial\Sigma} \mapsto \Omega(z) g\big|_{\partial\Sigma} \bar{\Omega}(z)^{-1}\,.$$

It is easy to envisage submanifolds $B \subset G$ which preserve some of the affine symmetry; although proving the conformal invariance of the corresponding boundary conditions might be much more difficult. We will concentrate here on boundary conditions such that $B$ is a (twisted) conjugacy class. The conformal invariance of boundary conditions associated to conjugacy classes has been shown, using various methods and in a nondecreasing amount of generalisation in the series of papers [1, 6, 5, 2].

### 2.1 Conjugacy classes

Let $G$ be a connected Lie group and $g_0 \in G$ an element. Let $C(g_0)$ denote the conjugacy class of the element $g_0$, defined as the subset of $G$ with the following elements:

$$C(g_0) := \left\{ g g_0 g^{-1} \mid g \in G \right\}\,.$$

The conjugacy class of an element $g_0$ is therefore the orbit of that element under the adjoint action of the group: $\mathrm{Ad}_g : G \to G$, defined by $\mathrm{Ad}_g(g_0) = g g_0 g^{-1}$. Each conjugacy class is a connected submanifold of $G$. Since every element $g_0$ belongs to one and only one conjugacy class, $G$ is foliated by its conjugacy classes. The leaves of the foliation need not all have the same topology; in other words, the foliation need not be a fibration.

For compact simple groups a well-known result says that every element is conjugate to some maximal torus $\mathbb{T}$. The Weyl group $W = N(\mathbb{T})/\mathbb{T}$ further relates elements of the maximal torus. Therefore the conjugacy classes are parametrised by the quotient $\mathbb{T}/W$ of a maximal torus by the Weyl group. For example, for $G = \mathrm{SU}(2)$, the conjugacy classes are parametrised by $S^1/\mathbb{Z}_2$, which we can understand as the interval $\theta \in [0, \pi]$. The conjugacy classes corresponding to $\theta = 0, \pi$ are points, corresponding to the elements in the centre of $\mathrm{SU}(2)$: $\pm\mathbb{1}$, whereas the classes corresponding to $\theta \in (0, \pi)$ are spheres. If we picture $\mathrm{SU}(2)$, which is homeomorphic to the 3-sphere, as the one-point compactification of $\mathbb{R}^3$ where the sphere at infinity is collapsed to a point, the foliation of $\mathrm{SU}(2)$ by its conjugacy classes coincides with the standard foliation of $\mathbb{R}^3$ by 2-spheres with two degenerate spheres at the origin and at infinity. Because of the degeneration of the limiting spheres the foliation is not a fibration. This is not surprising, since $S^3$ is a circle bundle over $S^2$ and not a sphere bundle over $S^1$.

For noncompact groups the problem is more subtle since there is no longer a notion of maximal torus, even in the semisimple case. In the absence of a general theory, one has to treat each case independently. For example, the conjugacy classes of $\mathrm{SL}(2, \mathbb{R})$ can be found in [6], in the context of $D$-branes on $AdS_3$. In this paper we will determine the conjugacy classes of the noncompact nonreductive Lie group $\mathcal{N}$ introduced by Nappi and Witten in [4] and which we term the Nappi-Witten group.

The boundary conditions associated with conjugacy classes preserve an infinite-dimensional affine symmetry

$$g\big|_{\partial\Sigma} \mapsto \Omega(z) g\big|_{\partial\Sigma} \bar{\Omega}(z)^{-1}\,, \tag{2.1}$$

where now $\bar{\Omega} = \Omega$ at the boundary, but otherwise arbitrary.

A similar situation obtains in the case of boundary conditions which say that $B \subset G$ is a shifted conjugacy class: $B = h\,C$ or $B = C\,h$, where $h \in G$ and $C$ is a conjugacy class. This also respects the affine symmetry in (2.1), but acting on $h^{-1}g$ and $gh^{-1}$ respectively.

## 2.2 Twisted conjugacy classes

A way to generalise this situation is to consider twisted conjugacy classes. Every automorphism $r$ of the group gives rise to a twisted version of conjugacy classes, by considering the orbit of a group element $h$, say, under the twisted conjugation $h \mapsto ghr(g)^{-1}$. If the automorphism is inner, so that there is a group element $k$ such that $r(g) = kgk^{-1}$, then the orbit under the twisted conjugation is simply the shifted conjugacy class $k^{-1}C(gk)$; however when the automorphism is not inner, the orbits are quite different, as we will see below in the case of the Nappi-Witten group. These boundary conditions are also preserved by the infinite-dimensional affine symmetry given by (2.1), but where now $\bar{\Omega} = r(\Omega)$ at the boundary.

More generally two automorphisms which are related by an inner automorphism give rise to twisted conjugacy classes which are translated relative to each other:

$$C_{r\circ \mathrm{Ad}_h}(g) = C_r(gr(h))r(h)^{-1} \qquad \text{and} \qquad C_{\mathrm{Ad}_h \circ r}(g) = C_r(gh)h^{-1}\,.$$

This suggests that we organise the possible boundary conditions according to the group of (metric-preserving) outer automorphisms. Indeed, let $\mathrm{Aut}_o(G)$ denote the group of metric-preserving automorphisms of $G$ and let $\mathrm{Inn}_o(G) \subset \mathrm{Aut}_o(G)$ denote the invariant subgroup corresponding to those automorphisms which are inner. Then one can define the factor group

$$\mathrm{Out}_o(G) := \mathrm{Aut}_o(G)/\mathrm{Inn}_o(G)\,,$$

of metric-preserving outer automorphisms. Elements of $\mathrm{Out}_o(G)$ are equivalence classes of metric-preserving automorphisms of $G$: two such automorphisms being equivalent if they are related by an inner automorphism. To each element of $\mathrm{Out}_o(G)$ we can associate an equivalence classes of $D$-branes foliating $G$, two such branes being equivalent if one is simply a translate of another. For example, corresponding to the identity in $\mathrm{Out}_o(G)$ we have conjugacy classes and their translates. Since $\mathrm{Out}_o(G)$ is trivial for a simple group, these are all the $D$-branes one expects to find associated to D-type boundary conditions. On the other hand, for abelian groups, no nontrivial automorphism is inner. In the case of the Nappi-Witten group, which we treat in detail in the next section, we have that $\mathrm{Out}_o(G) \cong \mathbb{Z}_2$, whence there will be two distinct families of $D$-branes, as we will see below.

It is tempting to interpret the group of metric-preserving outer automorphisms as a kind of duality group in the WZW model, permuting different types of $D$-branes much in the same way as T-duality or mirror symmetry in toroidal and Calabi-Yau compactifications, respectively.

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## 3. $D$-branes in the Nappi-Witten group

In this section we determine the $D$-branes in the Nappi-Witten group corresponding to (twisted) conjugacy classes. We start with some general remarks about the Nappi-Witten group we will make use of in the sequel.

### 3.1 The Nappi-Witten group

The Nappi-Witten group $\mathcal{N}$ is the universal central extension of the two-dimensional euclidean group. Topologically $\mathcal{N} \cong S^1 \times \mathbb{R}^3$, although one sometimes also considers its universal cover $\tilde{\mathcal{N}} \cong \mathbb{R}^4$.

It is therefore convenient to parametrise $\mathcal{N}$ by a triple $(\theta, w, t)$ where $\theta$ is an angle, $w$ is a complex number and $t$ is a real number. A typical group element shall be denoted $g(\theta, w, t)$. Our first task is to write down the group multiplication law.

We start with the two-dimensional euclidean group $\mathcal{E}$, with elements $h(\theta, w)$. It acts on the complex plane, parametrised by $z$, via affine transformations:

$$h(\theta, w) \cdot z = e^{i\theta} z + w \,.$$

From this action we can read off the group multiplication law:

$$h(\theta_1, w_1)\, h(\theta_2, w_2) = h(\theta_1 + \theta_2, w_1 + e^{i\theta_1} w_2) \,,$$

which displays the semi-direct product nature of the group. If we denote by $R(\theta)$ the rotation by an angle $\theta$ in the complex plane: $R(\theta) \cdot z = e^{i\theta} z$ and $T(w)$ the translation by $w$: $T(w) \cdot z = z + w$, then clearly $h(\theta, w) = T(w)\, R(\theta)$, and the group multiplication law above says, among other things, that

$$R(\theta_1)\, R(\theta_2) = R(\theta_1 + \theta_2) \tag{3.1}$$

$$R(\theta)\, T(w) = T(e^{i\theta} w)\, R(\theta) \tag{3.2}$$

Since the group $\mathcal{N}$ is a central extension of $\mathcal{E}$, a representation of $\mathcal{N}$ is a projective representation of $\mathcal{E}$. Such a representation is characterised by a group cocycle. In the present case, the cocycle is associated with the translations, which as a result no longer commute:

$$T(w_1)\, T(w_2) = T(w_1 + w_2)\, e^{-\frac{1}{2}\mathrm{Im}(w_1 \bar{w}_2)} \,.$$

(The normalisation has been chosen for later convenience.) Let us introduce an abstract one-parameter subgroup $Z(t)$ which acts as $e^t$ in the above projective representation. Then we can rewrite the above equation as

$$T(w_1)\, T(w_2) = T(w_1 + w_2)\, Z\left(-\frac{1}{2}\mathrm{Im}(w_1 \bar{w}_2)\right) . \tag{3.3}$$

Let $g(\theta, w, t) = T(w)\, R(\theta)\, Z(t)$ denote an element of $\mathcal{N}$. Then the group multiplication can be read off from equations (3.1), (3.2) and (3.3) to give:

$$g(\theta_1, w_1, t_1)\, g(\theta_2, w_2, t_2) = g\left(\theta_1 + \theta_2, w_1 + e^{i\theta_1} w_2, t_1 + t_2 - \frac{1}{2}\mathrm{Im}(w_1 e^{-i\theta_1} \bar{w}_2)\right) . \tag{3.4}$$

It follows from that $g(0,0,0)$ is the identity element and that the inverse of $g(\theta, w, t)$ is given by

$$g(\theta, w, t)^{-1} = g(-\theta, -e^{-i\theta} w, -t) \,. \tag{3.5}$$

### 3.2 The Nappi-Witten Lie algebra

In order to make contact with the traditional approach to the WZW model, where the fundamental dynamical variables are the chiral currents, we must discuss the Lie algebra $\mathfrak{n}$ of the Nappi-Witten Lie group $\mathcal{N}$.

Let us introduce abstract generators $\mathsf{P}_1$, $\mathsf{P}_2$, $\mathsf{J}$ and $\mathsf{K}$ for the Lie algebra $\mathfrak{n}$ of $\mathcal{N}$. We postulate the following relations between these generators and the group elements $T(w)$, $R(\theta)$ and $Z(t)$:

$$T(w) = \exp(x\mathsf{P}_1 + y\mathsf{P}_2)\,, \qquad R(\theta) = \exp(\theta\mathsf{J}) \qquad \text{and} \qquad Z(t) = \exp(t\mathsf{K})\,, \tag{3.6}$$

where $w = x + iy$. From the equations (3.1), (3.2) and (3.3) one can easily work out the Lie brackets obeyed by these generators. One finds

$$[\mathsf{J}, \mathsf{P}_i] = \epsilon_{ij}\mathsf{P}_j\,, \qquad \text{and} \qquad [\mathsf{P}_i, \mathsf{P}_j] = \epsilon_{ij}\mathsf{K}\,, \tag{3.7}$$

with all other brackets vanishing. As is well known by now, this algebra, although solvable, possesses an invariant scalar product with lorentzian signature:

$$\langle \mathsf{P}_i, \mathsf{P}_j \rangle = \delta_{ij} \qquad \text{and} \qquad \langle \mathsf{J}, \mathsf{K} \rangle = 1\ ; \tag{3.8}$$

and this means that the group $\mathcal{N}$ possesses a bi-invariant lorentzian metric.

We will need the metric in order to determine the geometry of the $D$-branes, so we compute it now. By definition, the metric is given by

$$ds^2(g) = \langle g^{-1}dg, g^{-1}dg \rangle\,,$$

where $g^{-1}dg$ denotes the left-invariant Maurer-Cartan $\mathfrak{n}$-valued one-forms, and $\langle -, - \rangle$ is the scalar product in $\mathfrak{n}$ given by equation (3.8).

A simple calculation reveals that

$$\begin{aligned} g^{-1}dg = &(\cos\theta dx + \sin\theta dy)\mathsf{P}_1 + (\cos\theta dy - \sin\theta dx)\mathsf{P}_2 + \\ &+ d\theta\mathsf{J} + \left(dt - \frac{1}{2}xdy + \frac{1}{2}ydx\right)\mathsf{K}\,, \end{aligned} \tag{3.9}$$

whence the metric is given by

$$ds^2 = dx^2 + dy^2 + 2d\theta dt + (ydx - xdy)d\theta\,. \tag{3.10}$$

Bi-invariance is easily verified by showing that $ds^2 = \langle dgg^{-1}, dgg^{-1} \rangle$.

Let us remark that one can additionally set $\langle \mathsf{J}, \mathsf{J} \rangle$ equal to any real number and still have an invariant scalar product, but there exists a Lie algebra automorphism which sets it back to zero. Therefore it represents no real loss in generality to demand that it be zero from the outset. In any case, its inclusion would simply add a term $bd\theta^2$ to the metric (3.10), where $b$ is a real constant. None of the results we obtain are changed in any qualitative way by the inclusion of this constant.



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## 3.3 Automorphisms of the Nappi-Witten group

In order to determine the different types of $D$-branes in the Nappi-Witten group, we must determine the metric-preserving automorphisms of the Nappi-Witten group $\mathcal{N}$. We will spare the reader the routine calculation by which one determines these automorphisms, and sketch the calculation instead. We work with the Lie algebra: automorphisms of the Lie group and the Lie algebra are related by the exponential map, which is a diffeomorphism in this case. Since the Lie algebra $\mathfrak{n}$ is four-dimensional with a minkowskian scalar product, the automorphisms of $\mathfrak{n}$ which preserve the scalar product belong to $\mathrm{O}(3,1)$. We will refer to them as *orthogonal automorphisms*. Determining the group of orthogonal automorphisms of $\mathfrak{n}$ is thus a linear algebra problem, which can be easily solved to yield the following.

The orthogonal automorphisms of $\mathfrak{n}$ are parametrised by $\mathbb{R}^2 \times S^1 \times \mathbb{Z}_2$. Indeed suppose that $(x, y) \in \mathbb{R}^2$, $\theta \in S^1$ and $\varepsilon = \pm 1$. Then the automorphism $R$ corresponding to $(x, y, \theta, \varepsilon)$ is given by

$$\begin{aligned} R(\mathsf{P}_1) &= \varepsilon(\cos\theta \mathsf{P}_1 - \sin\theta \mathsf{P}_2) + (y\sin\theta - x\cos\theta)\mathsf{K} \\ R(\mathsf{P}_2) &= \sin\theta \mathsf{P}_1 + \cos\theta \mathsf{P}_2 - \varepsilon(x\sin\theta + y\cos\theta)\mathsf{K} \\ R(\mathsf{J}) &= x\mathsf{P}_1 + y\mathsf{P}_2 + \varepsilon\mathsf{J} - \frac{1}{2}\varepsilon(x^2+y^2)\mathsf{K} \\ R(\mathsf{K}) &= \varepsilon\mathsf{K}\,. \end{aligned}$$

It is easy to show that automorphisms for which $\varepsilon = +1$ are inner, whence the group of orthogonal outer automorphisms is isomorphic to $\mathbb{Z}_2$. This means that there will be two distinct families of $D$-branes: (the translates of) conjugacy classes, and another family corresponding to (the translates of) twisted conjugacy classes. In order to determine these twisted conjugacy classes, it will prove sufficient to study just one automorphism which is not inner. The simplest such automorphism, $R$, is the one corresponding to $x = y = \theta = 0$ and $\varepsilon = -1$:

$$R(\mathsf{P}_1) = -\mathsf{P}_1\,, \qquad R(\mathsf{P}_2) = \mathsf{P}_2\,, \qquad R(\mathsf{J}) = -\mathsf{J}\,, \qquad R(\mathsf{K}) = -\mathsf{K}\,.$$

The induced automorphism $r$ on the Lie group is defined by:

$$r : g(\theta, w, t) \mapsto g(-\theta, -\bar{w}, -t)\,. \tag{3.11}$$

It is easy to check that this is an automorphism which moreover is not inner, since as we will see below, conjugation leaves $\theta$ invariant.

We now proceed to discuss the two families of $D$-branes.

## 3.4 Conjugacy classes as orbits of the adjoint group

The first family of $D$-branes is the one associated to the identity automorphism, which we have seen give rise to conjugacy classes.



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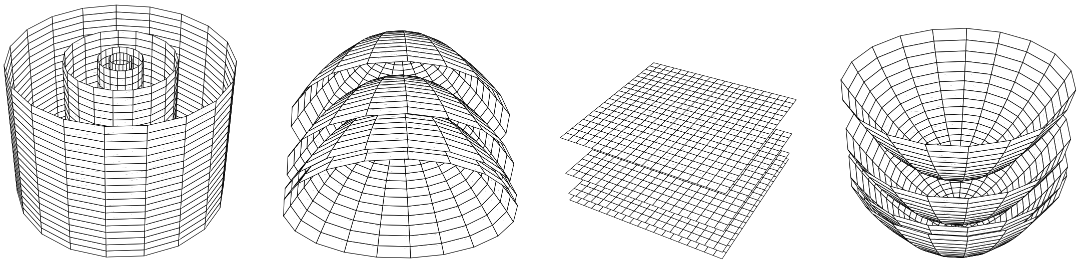
| Class | Element | Type | Centraliser | Causal Type |
|---|---|---|---|---|
| $\mathcal{C}'_{0,t_0}$ | $g(0,0,t_0)$ | point | $\mathcal{N}$ | |
| $\mathcal{C}_{0,\lvert w_0\rvert}$ | $g(0,w_0,0)$ | cylinder | $g(0,sw_0,t)$ | degenerate |
| $\mathcal{C}_{\pi,t_0}$ | $g(\pi,0,t_0)$ | plane | $g(\theta,0,t)$ | spacelike |
| $\mathcal{C}_{\theta_0\neq 0,\pi,\ k_0}$ | $g(\theta_0,0,k_0)$ | paraboloid | $g(\theta,0,t)$ | spacelike |

**Table 1:** Conjugacy classes of the Nappi-Witten group, along with the typical element, the topological type of the conjugacy class, the centraliser of the typical element and the causal type of the class.



$\theta_0 = 0$ $\qquad$ $0 < \theta_0 < \pi$ $\qquad$ $\theta_0 = \pi$ $\qquad$ $2\pi > \theta_0 > \pi$

**Figure 1:** Foliation of $\theta_0 =$ constant three-planes by conjugacy classes.

Using the explicit expression (3.5) for the inverse of a group element and the group multiplication law (3.4), we can compute the adjoint action of a generic element $g(\theta, w, t)$ on a fixed element $g(\theta_0, w_0, t_0)$, defined by

$$\mathrm{Ad}_{g(\theta,w,t)}\, g(\theta_0, w_0, t_0) := g(\theta, w, t)\, g(\theta_0, w_0, t_0)\, g(\theta, w, t)^{-1}\,.$$

We find

$$\begin{aligned}&\mathrm{Ad}_{g(\theta,w,t)}\, g(\theta_0, w_0, t_0) = \\ &= g\left(\theta_0, (1-e^{i\theta_0})\, w + e^{i\theta}\, w_0, t_0 - \frac{1}{2}|w|^2 \sin\theta_0 - \frac{1}{2}\mathrm{Im}(1+e^{i\theta_0}) w e^{-i\theta} \bar{w}_0\right). \qquad (3.12)\end{aligned}$$

The first thing we notice is that $\theta_0$ is an invariant of the conjugacy class, and we can try to distinguish between the different conjugacy classes by the value of $\theta_0$ in the first place. The three-plane defined by fixing the value of $\theta$ is foliated by the conjugacy classes. The results are summarised in table 1 and illustrated in figure 1.

Consider first of all the case of $\theta_0 = 0$. Then we see that under the adjoint action (3.12),

$$g(0, w_0, t_0) \mapsto g\left(0, e^{i\theta}\, w_0, t_0 - \mathrm{Im} w e^{-i\theta} \bar{w}_0\right).$$

We can distinguish two subcases:

- ($w_0 = 0$) In this case, $g(0, 0, t_0)$ belongs to the centre of $\mathcal{N}$ and hence is the only element in its conjugacy class $\mathcal{C}'_{0,t_0}$. The centraliser of every such class is the group $\mathcal{N}$ itself.

- ($w_0 \neq 0$) In this case, the conjugacy class is a cylinder (i.e. diffeomorphic to $\mathbb{R}\times S^1$) comprising those group elements of the form $g(0, w_0 e^{i\theta}, t)$, which shows that the conjugacy class is labelled by $|w_0|$. We denote this class by $\mathcal{C}_{0,|w_0|}$. The centraliser of a typical element $g(0, w_0, 0)$ in this class is the two-dimensional abelian subgroup $\mathcal{A}_{w_0} \subset \mathcal{N}$ whose elements are of the form $g(0, sw_0, t)$, with $s$ and $t$ real numbers.

Figure 1$a$ shows how these conjugacy classes foliate the three-plane in $\mathcal{N}$ corresponding to $\theta_0 = 0$.

Consider now $\theta_0 = \pi$. In this case, under the adjoint action (3.12),

$$g(\pi, w_0, t_0) \mapsto g(\pi, 2w + e^{i\theta} w_0, t_0)\,.$$

This shows that the conjugacy class is a 2-plane labelled by $t_0$, denoted $\mathcal{C}_{\pi,t_0}$ and pictured in figure 1$c$. The centraliser of an element $g(\pi, w_0, t_0)$ in $\mathcal{C}_{\pi,t_0}$ is the two-parameter subgroup $\mathcal{S}_{w_0} \subset \mathcal{N}$ consisting of elements of the form

$$g\left(\theta, \frac{1}{2}(1 - e^{i\theta}) w_0, t\right).$$

Topologically, $\mathcal{S}_{w_0} \cong \mathbb{R} \times S^1$; although as Lie groups it is not a product. Indeed, if we let $k(\theta, t) := g(\theta, \frac{1}{2}(1 - e^{i\theta}) w_0, t)$, then we have

$$k(\theta_1, t_1)\, k(\theta_2, t_2) = k\left(\theta_1 + \theta_2, t_1 + t_2 - \frac{1}{8}|w_0|^2 \mathrm{Im}(1 - e^{i\theta_1})(1 - e^{i\theta_2})\right).$$

Finally we consider the case of general $\theta_0 \neq 0, \pi$. The centraliser of an element $g(\theta_0, w_0, t_0)$ with $\theta_0 \neq 0, \pi$ is given by those elements of the form

$$g(\theta, (1 - e^{i\theta}) w_0 (1 - e^{i\theta_0})^{-1}, t)\,,$$

which form a two-dimensional subgroup $\mathcal{S}_{z_0} \subset \mathcal{N}$ with $z_0 := w_0(1 - e^{i\theta_0})^{-1}$. The conjugacy class, being diffeomorphic to $\mathcal{N}/\mathcal{S}_{z_0}$, is therefore two-dimensional. The conjugacy class is defined (having fixed $\theta_0$) by the value of the real-valued class function

$$F(\theta_0, w_0, t_0) := t_0 \sin\frac{\theta_0}{2} + \frac{1}{4}|w_0|^2 \cos\frac{\theta_0}{2}\,.$$

Fixing a real number $k_0$ and $\theta_0 \neq 0, \pi$, elements in the conjugacy class $\mathcal{C}_{\theta_0,k_0}$ are of the form

$$g\left(\theta_0, w, k_0 - \frac{1}{4}|w|^2 \cot\frac{\theta_0}{2}\right),$$

for $w \in \mathbb{C}$, which make up a paraboloid. We can distinguish two cases: $\theta_0 \in (0, \pi)$ and $\theta_0 \in (\pi, 2\pi)$. They are illustrated in figures 1$b$ and 1$d$, respectively.

Parenthetically, the function $F$ also distinguishes conjugacy classes when $\theta_0 = \pi$ and even for $\theta_0 = 0$ when $|w_0| \neq 0$; whereas for $w_0 = 0$, it is the value of $t_0$ which now distinguishes the (pointlike) conjugacy classes.

### 3.5 Geometry of the conjugacy classes

Due to their interpretation as $D$-branes, an important characteristic of a conjugacy class is its geometry. In particular we have seen above that in order to conclude that a conjugacy class may serve as Dirichlet boundary conditions for a WZW model, it was necessary to assume that it was non-degenerate relative to the bi-invariant metric. In this section we will elucidate the geometry of the conjugacy classes found above. We will see that the planar and paraboloidal conjugacy classes are flat and euclidean, whereas the cylindrical conjugacy class is degenerate. Therefore all but the cylindrical conjugacy classes can be understood as $D$-branes, at least at our current level of understanding. Since the metric on $\mathcal{N}$ is bi-invariant, the calculation is simplified in that it is enough to determine the geometry at a point: all other points being related by the adjoint action of the group, which is an isometry. At the same time, left- or right-translating a conjugacy class does not alter its geometry.

As alluded to above, the bi-invariant metric on $\mathcal{N}$ is not unique (even up to scale): one can always add a term $bd\theta^2$ for $b \in \mathbb{R}$. Since, as we shall see, the conjugacy classes have constant $\theta$, the induced metric is impervious to the inclusion of this term.

#### 3.5.1 The conjugacy classes $\mathcal{C}_{0,|w_0|}$

From the above considerations it is enough to work at a point, which we take to be the typical element $g_0 := g(0, w_0, 0)$. The conjugacy class is the set $\mathcal{C} = \{g(0, e^{i\theta}w_0, t) \mid \theta, t \in \mathbb{R}\}$, and the centraliser is the subgroup $\mathcal{Z} = \{g(0, sw_0, t) \mid s, t \in \mathbb{R}\}$. At the point $g_0$, the tangent space $T_{g_0}\mathcal{C}$ to the conjugacy class is spanned by the vectors $v_1 = x_0\partial_y - y_0\partial_x$ and $v_2 = \partial_t$, where $w_0 = x_0 + iy_0$. Computing their scalar product, we find that at the point $g(0, w_0, 0)$,

$$\langle v_1, v_1\rangle = |w_0|^2 \,, \langle v_1, v_2\rangle = \langle v_2, v_2\rangle = 0\,.$$

Therefore the metric is degenerate, and this means that this conjugacy class cannot be interpreted (at least straightforwardly) as a $D$-brane. Notice that the tangent space $T_{g_0}\mathcal{Z}$ to the centraliser $\mathcal{Z}$ at the point $g_0$, is spanned by the vectors $u_1 = x_0\partial_x + y_0\partial_y$ and $u_2 = \partial_t$, which obey

$$\langle u_1, u_1\rangle = |w_0|^2 \,, \langle u_1, u_2\rangle = \langle u_2, u_2\rangle = 0\,.$$

As usual we find that at $g_0$, $T_{g_0}\mathcal{C} = (T_{g_0}\mathcal{Z})^\perp$, but that in this case $T_{g_0}\mathcal{C} \cap T_{g_0}\mathcal{Z} \neq \{0\}$.

#### 3.5.2 The conjugacy classes $\mathcal{C}_{\pi,t_0}$

Let us now fix the typical element $g_0 := g(\pi, 0, t_0)$. The conjugacy class of this element is the set $\mathcal{C} = \{g(\pi, w, t_0) \mid w \in \mathbb{C}\}$ and the normaliser of the typical element is the subgroup $\mathcal{Z} = \{g(\pi + \theta, 0, t_0 + t) \mid \theta, t \in \mathbb{R}\}$. At the point $g_0$, the tangent space to the conjugacy class is spanned by the vectors $v_1 = \partial_x$ and $v_2 = \partial_y$, with metric

$$\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = 1\,, \langle v_1, v_2\rangle = 0\,.$$

This metric is clearly non-degenerate and euclidean. It is also evidently flat. Therefore the resulting $D$-brane is a flat euclidean $D$-string. The tangent space to the centraliser subgroup at the point $g_0$ is spanned by $u_1 = \partial_\theta$ and $u_2 = \partial_t$, whose metric is

$$\langle u_1, u_2\rangle = 1\,, \langle u_1, u_1\rangle = \langle u_2, u_2\rangle = 0\,,$$

which is non-degenerate and minkowskian. Again we have that $T_{g_0}\mathcal{C} = (T_{g_0}\mathcal{Z})^\perp$, but now $T_{g_0}\mathcal{C} \cap T_{g_0}\mathcal{Z} = \{0\}$ , so that $T_{g_0}\mathcal{N} = T_{g_0}\mathcal{C} \oplus T_{g_0}\mathcal{Z}$.

### 3.5.3 The conjugacy classes $\mathcal{C}_{\theta_0,k_0}$

We take $g_0 := g(\theta_0, 0, k_0)$ as typical element. Its conjugacy class is the set $\mathcal{C} = \{g(\theta_0, w, k_0 - \frac{1}{4}|w|^2 \cot\frac{\theta_0}{2}) \mid w \in \mathbb{C}\}$, and its centraliser subgroup is $\mathcal{Z} = \{g(\theta, 0, t) \mid \theta, t \in \mathbb{R}\}$. The tangent space to the conjugacy class at $g_0$ is spanned by the vectors $v_1 = \partial_x$ and $v_2 = \partial_y$, with metric

$$\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = 1\,, \langle v_1, v_2\rangle = 0\,.$$

This metric is clearly non-degenerate, euclidean and flat. Therefore the resulting $D$-brane is again a flat euclidean $D$-string. The tangent space to the centraliser subgroup at the point $g_0$ is spanned by $u_1 = \partial_\theta$ and $u_2 = \partial_t$, whose metric is

$$\langle u_1, u_2\rangle = 1\,, \langle u_1, u_1\rangle = \langle u_2, u_2\rangle = 0\,,$$

which is non-degenerate and minkowskian. Again we have that $T_{g_0}\mathcal{C} = (T_{g_0}\mathcal{Z})^\perp$, and $T_{g_0}\mathcal{C} \cap T_{g_0}\mathcal{Z} = \{0\}$ , so that $T_{g_0}\mathcal{N} = T_{g_0}\mathcal{C} \oplus T_{g_0}\mathcal{Z}$.

## 3.6 Twisted conjugacy classes of the Nappi-Witten group

In this section we determine the twisted conjugacy classes in the Nappi-Witten group. As discussed above, they will all be translates of the twisted conjugacy classes corresponding to the automorphism $r$ defined in equation (3.11), which is a representative for the unique nontrivial (metric-preserving) outer automorphism in the Nappi-Witten group.

Thus let $C_r(\theta_0, w_0, t_0)$ denote the twisted adjoint orbit of the element $g(\theta_0, w_0, t_0)$:

$$C_r(\theta_0, w_0, t_0) := \left\{g(\theta, w, t)g(\theta_0, w_0, t_0)g(-\theta, -\bar{w}, -t)^{-1} \mid (\theta, w, t) \in S^1 \times \mathbb{C} \times \mathbb{R}\right\}.$$

Using the group multiplication law (3.4), one can compute the twisted action. First of all notice that

$$g(-\theta, -\bar{w}, -t)^{-1} = g(\theta, e^{i\theta}\bar{w}, t)\,,$$

and therefore

$$g(\theta,w,t)g(\theta_0,w_0,t_0)g(\theta,e^{i\theta}\bar{w},t) = g\big(\theta_0+2\theta, e^{i\theta}w_0 + w + e^{i(2\theta+\theta_0)}\bar{w},$$
$$t_0 + 2t - \frac{1}{2}\mathrm{Im}(we^{-i\theta}\bar{w}_0) - \frac{1}{2}\mathrm{Im}((w+e^{i\theta}w_0)e^{-i(\theta_0+2\theta)}w)\big)\,. \quad (3.13)$$

In order to determine the dimension of this orbit, we work out the isotropy of the point $(\theta_0, w_0, t_0)$; that is, the subgroup of $\mathcal{N}$ whose twisted adjoint action leaves $g(\theta_0, w_0, t_0)$ invariant. Equating the right-hand side of equation (3.13) with $g(\theta_0, w_0, t_0)$ we find that $\theta = 0$, $w = i\varrho e^{i\theta_0/2}$, and $t = \frac{1}{2}\varrho\mathrm{Re}(w_0 e^{-i\theta_0/2})$, for some real parameter $\varrho$. In other words, the isotropy subgroup is one-dimensional and hence the twisted conjugacy classes are three-dimensional.

Since they have codimension one, it is possible to exhibit these twisted conjugacy classes as level sets of a function which is invariant under the twisted adjoint action. The infinitesimal generators of the twisted adjoint action are easy to work out and demanding that a function be invariant under their action gives a number of partial differential relations. The differential ring of functions satisfying these relations is generated by the function

$$G(\theta_0, w_0, t_0) = \mathrm{Im}(w_0 e^{-i\theta_0/2})\,.$$

Hence each twisted conjugacy class is determined by a real number $k$, as the set of $g(\theta, w, t)$ for which

$$\mathrm{Im}(we^{-i\theta/2}) = k\,. \quad (3.14)$$

It may be worth remarking that the twisted conjugacy classes in this family are odd-dimensional. This is in sharp contrast with the case of conjugacy classes, which (at least in groups admitting bi-invariant metrics) always have even dimension. This follows because conjugacy classes are the image under the exponential map of the adjoint orbits, which are diffeomorphic to the co-adjoint orbits which are symplectic manifolds relative to the natural Kirillov-Kostant-Souriau symplectic structure, and hence are even-dimensional.

### 3.7 Geometry of the twisted conjugacy classes

The induced metric on the twisted conjugacy classes can be worked out as follows. Let us introduce polar coordinates $(r, \phi)$ related to $w$ in the usual way: $w = re^{i\phi}$. Let us assume that $k \neq 0$. We can then eliminate the radial coordinate $r$ using equation (3.14):

$$r = k \csc\left(\phi - \frac{1}{2}\theta\right)\,.$$

It is convenient to eliminate $\phi$ in terms of the real variable $z = \cot(\phi - \frac{1}{2}\theta)$. In terms of the coordinates $(z, \theta, t)$ parametrising the twisted conjugacy class, the metric (3.10) becomes

$$ds^2 = k^2 dz^2 + 2d\theta \left( dt - \frac{1}{8} k^2 (1 + z^2) d\theta \right) , \tag{3.15}$$

which is clearly non-degenerate and of lorentzian signature. Therefore twisted conjugacy classes define $D$-membranes. Unlike the euclidean $D$-strings associated to the conjugacy classes, these membranes are not flat and, since in three dimensions, Ricci-flatness implies flatness, they are not Ricci-flat either. To see this let us simply notice that the riemannian connection is given by

$$\nabla_\theta \partial_\theta = \frac{1}{4} z \partial_z \qquad \text{and} \qquad \nabla_\theta \partial_z = \nabla_z \partial_\theta = -\frac{1}{4} k^2 z \partial_t \,,$$

whence the riemannian curvature tensor has components

$$R_{z\theta\theta}{}^{z} = -\frac{1}{4} \qquad \text{and} \qquad R_{z\theta z}{}^{t} = \frac{1}{4} k^2 \,.$$

Notice that the Ricci tensor has non-vanishing components $R_{\theta\theta} = 1/4$, but that the scalar curvature does vanish.

Let us remark that the membrane admits parallel spinors. In fact, it is an example of an indecomposable Ricci-null lorentzian manifold (see for example [3]) with holonomy group isomorphic to $\mathbb{R} \subset \mathrm{SO}(2,1)$. In fact the membrane metric conforms to the metric given by equation (17) in [3], which is the most general three-dimensional lorentzian metric admitting parallel spinors.

The same conclusion can be reached for $k = 0$; although the details are different. In this case, we do not eliminate $r$, but rather notice that in this case $\phi = 2\theta$, whence in terms of the real coordinates $(r, \theta, t)$, the induced metric on the membrane is given by

$$ds^2 = dr^2 + 2d\theta \left( dt - \frac{3}{8} r^2 d\theta \right) , \tag{3.16}$$

which again describes an indecomposable three-dimensional lorentzian manifold admitting parallel spinors.

Let us remark that the induced metric on the twisted conjugacy classes is affected by including the term $bd\theta^2$ coming from the ambiguity in the invariant scalar product $\mathfrak{n}$; but this does not change qualitatively the geometry of the brane. One still obtains indecomposable three-dimensional lorentzian manifolds admitting parallel spinors.

In summary, we see that the Nappi-Witten group admits curved membranes admitting parallel spinors, flat euclidean strings and instantons among its $D$-branes.



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## 4. Conclusions

In this paper we have re-examined the possible boundary conditions in a WZW model which are compatible with conformal invariance. We have shown (following [5]) how to relate gluing conditions on the chiral currents to geometric boundary conditions of the group valued fields in the classical description of the WZW model. We have seen that a natural solution to the consistency conditions are given by (twisted) conjugacy classes: the orbits of group elements under the adjoint action twisted by an automorphism. When the automorphism is trivial, the resulting orbits are conjugacy classes, and when it is inner the orbits are shifted conjugacy classes. More generally, two automorphisms whose "difference" is an inner automorphism give rise to orbits which are shifted relative to each other. This suggests that one should use the group of (metric-preserving) outer automorphisms as a sort of classifying group for boundary conditions.

It is important to keep in mind that not all boundary conditions are given by (twisted) conjugacy classes, as shown in [7, 6, 5].

We have then illustrated these results in the case of the Nappi-Witten group. In that case the group of (metric-preserving) outer automorphisms has order 2, and hence there are two distinct families of possible boundary conditions. One family gives rise to (shifted) conjugacy classes. Studying their geometry, we have seen that these classes consist of flat euclidean $D$-strings and $D$-instantons. The other family of boundary conditions gives rise to $D$-membranes which are not flat: they are indecomposable lorentzian manifolds admitting parallel spinors.



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